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Assignment IV: MTH 213, Fall 2017

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QUESTION 1. (i) Prove that $\sqrt{5}$ is irrational

Solution: Deny. Then $\sqrt{5} = \frac{a}{b}$ **(note a, b must be odd (see class notes) and** $gcd(a,b) = 1$**). Hence as in class we have**

$5 = \frac{(2k+1)^2}{(2m+1)^2}$ **(note that** $k, m \in Z$**).**

so we have $20m^2 + 20m + 5 = 4k^2 + 4k + 1$

Hence $20m^2 + 20m + 4 = 4k^2 + 4k$**. Divide by 5 we have**

$5m^2 + 5m + 1 = k^2 + k$**, impossible since for any m we have** $5m^2 + 5m + 1$ **is an odd integer and for any** k **we have** $k^2 + k$ **is even.**

Thus $\sqrt{5}$ **is irrational.**

(ii) Prove $\sqrt{21}$ is irrational [hint: same argument as in (i), we conclude $?m^2 + ?m + 5$ is odd where $k^2 + k$ is even]

(iii) Prove $\sqrt{45}$ is irrational [Trivial $\sqrt{45} = 3\sqrt{5}$ done by (i)]

(iv) Prove $\sqrt{48}$ is irrational. [Trivial $\sqrt{48} = 4\sqrt{3}$ and we proved $\sqrt{3}$ is irrational]

(v) let n be an even number of the form $2m$ for some odd number m. Prove that $\sqrt{n}$ is irrational [hint: Deny. Then observe that $n = a/b$ where a must be even, b must be odd and of course gcd(a, b) = 1.]

Solution: Deny. Then $\sqrt{n} = \frac{a}{b}$ **(note that** a **must be even and** b **must be odd (see class notes) and** $gcd(a,b) = 1$**). Hence as in class we have**

$n = 2m = \frac{(4k^2}{(2w+1)^2}$ **(note that** $k, w \in Z$**).**

so we have $8mw^2 + 8mw + 2m = 4k^2$ **(note m is odd)**

divide by 4 we get $2mw^2 + 2mw + \frac{2m}{4} = k^2 \in Z$**, a contradiction since** $\frac{2m}{4}$ **is not an integer (because m is odd).**

Thus $\sqrt{n}$ **is irrational.**

(vi) Enough training. So in general let $n = p_1 p_2 \cdots p_k$ where the $p_i's$ are distinct odd prime numbers ($k \geq 1$) and assume that 4 is not a factor of $n - 1$. Prove that $\sqrt{n}$ is irrational (i.e., if n is a product of k distinct odd prime numbers and 4 is not a factor of $n - 1$, then $\sqrt{n}$ is irrational.)Note that as a special case, assume $k = 1$, then n is prime and hence it follows that if n is a prime number and 4 is not a factor of $n - 1$, then $\sqrt{n}$ is irrational.

Solution: Deny. Then $\sqrt{n} = \frac{a}{b}$ **(note that** a, b **are odd integers and** $gcd(a,b) = 1$**). Hence as in class we have**

$n = \frac{(2k+1)^2}{(2m+1)^2}$ **(note that** $k, m \in Z$**).**

so we have $4nm^2 + 4nm + n = 4k^2 + 4k + 1$

Hence $4nm^2 + 4nm + n - 1 = 4k^2 + 4k$**.**

divide by 4 we get $nm^2 + m + \frac{n-1}{4} = k^2 + k \in Z$**, a contradiction since** $\frac{n-1}{4}$ **is not an integer.**

Thus $\sqrt{n}$ **is irrational.**

(vii) Prove that $\sqrt{19}$ is irrational [see (vI)]

(viii) Prove that $\sqrt{87}$ is irrational[see (VI)]

(ix) Let p_1, p_2 be distinct prime numbers such that 4 is not a factor of $(p_1 p_2 - 1)$. Prove that $\sqrt{p_1} + \sqrt{p_2}$ is irrational.

Solution: Deny. Then $\sqrt{p_1} + \sqrt{p_2} = \frac{a}{b}$**, where** $gcd(a,b) = 1$.

Hence $(\sqrt{p_1} + \sqrt{p_2})^2 = \frac{a^2}{b^2}$

Thus $p_1 + 2\sqrt{p_1 p_2} + p_2 = \frac{a^2}{b^2}$**. Solve for** $\sqrt{p_1 p_2}$

We have $\sqrt{p_1 p_2} = \frac{a^2}{2b^2} - \frac{p_1}{2} - \frac{p_2}{2}$ **is a rational number, a contradiction, because** 4 **is not a factor of** $p_1 p_2 - 1$**, and thus** $\sqrt{p_1 p_2}$ **is irrational by (vi). Hence** $\sqrt{p_1} + \sqrt{p_2}$ **is irrational**

(x) Prove that $\sqrt{27} + \sqrt{13}$ is irrational. **Solution: Deny. Note** $\sqrt{27} + \sqrt{13} = 3\sqrt{3} + \sqrt{13}$**,** $p_1 = 3$ **and** $p_2 = 13$ **are prime numbers and 4 is not a factor of** $p_1 p_2 - 1 = 38$**. So use similar argument as in(vi). Deny**

$3\sqrt{3} + \sqrt{13} = \frac{a}{b}$**, where** $gcd(a,b) = 1$.

Hence $(3\sqrt{3} + \sqrt{13})^2 = \frac{a^2}{b^2}$

Thus $27 + 6\sqrt{39} + 13 = \frac{a^2}{b^2}$**. Solve for** $\sqrt{39}$

We have $\sqrt{39} = \frac{a^2}{2b^2} - \frac{27}{6} - \frac{13}{6}$ **is a rational number, a contradiction, because** 4 **is not a factor of** $p_1p_2 - 1 = 38$**, and thus** $\sqrt{39}$ **is irrational by (vi). Hence** $\sqrt{27} + \sqrt{13} = 3\sqrt{3} + \sqrt{13}$ **is irrational**

Remark: The method I presented here does not exist in the book or over the net, but this method does not work perfectly if 4 **is a factor of** $n - 1$**. I can use a different method to show that the above are still true when** 4 **is a factor of** $(n - 1)$**. The method involves a fact from number theory that says: If** $p_1p_2 \cdots p_k$ **are DISTINCT PRIME numbers and** $p_1p_2 \cdots p_k \mid n^2$**, then** $p_1p_2 \cdots p_k \mid n$**. In particular if** $k = 1$**, then if** $p_1 \mid n^2$**, then** $p_1 \mid n$ **(**p_1 **is prime). Note that in general if** m, n **are integers and** $m \mid n^2$**, then** m **need not be a factor of** n**. For example: let** $m = 8$**,** $n = 4$**. Then** m **is a factor of** n^2**, but** m **is not a factor of** n**. However, I do not want to use this method and I will try to develop the method I used here for the case** $4 \mid (n - 1)$**. If no success, then I will use this method later on**

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